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APPLICATION OF THE M-INTEGRAL TO CRACKED ANISOTROPIC COMPOSITE WEDGES

KUANG-CHONG WU
Institute of Applied Mechanics, National Taiwan University, Taipei, Taiwan 10764, R.O.C.

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Abstract—The path-independent M-integral is utilized to derive close form solutions for energy release rates for edge cracks in composite wedges under several loading conditions. The composite wedge considered consists of n wedges of different anisotropic elastic materials and wedge angles. The edge crack extends from the apex of the composite wedge and can lie either in the homogeneous region or along one of the interfaces. Three types of problem are considered. In the type 1 problem, concentrated forces are applied at the apexes. In the type 2 problem, the faces of the composite wedge are subjected to constant displacements. In the type 3 problem, mixed boundary conditions are prescribed at the wedge faces. For interface cracks, results are valid for "traction-free" cracks as well as for cracks with contact zones. For type 2 and type 3 problems, it is found that the energy release rate is independent of the crack orientation.

1. INTRODUCTION

Various authors (Eshelby, 1975; Freund, 1978; Ouchterlony, 1978) have given an approach based on the path-independent M-integral to calculate the energy release rates for some cracks in homogeneous isotropic elastic bodies. Nachman and Walton (1980) and Kubo (1982) showed that the approach could also be applied to interface cracks in isotropic composites. The case of an edge crack in an angularly inhomogeneous isotropic wedge has been studied by Wu and Chen (1989). In this paper, the approach is used to derive energy release rates for an edge crack in a composite wedge, as shown in Fig. 1. The composite wedge consists of n wedges of different anisotropic elastic materials. In the cylindrical coordinate system, each wedge occupies the region

$$\theta_{k-1} \leqslant \theta \leqslant \theta_k, \quad 0 \leqslant r \leqslant \infty, \quad k = 1, 2, \ldots, n \tag{1}$$

in which

$$\theta_0 < \theta_1 < \cdots < \theta_n. \tag{2}$$

The edge crack penetrates from the apex into the composite and is inclined by an angle θ_c with $\theta_0 < \theta_c < \theta_n$. The crack tip can either lodge at a homogeneous region or at one of the interfaces. Three types of problem are considered:

1. The apexes of the cracked wedge p^+ and p^- are loaded by concentrated forces.
2. The outer faces of the composite wedge, $\theta = \theta_0$ and $\theta = \theta_n$, are subjected to constant displacements.
3. The wedge faces are subjected to constant displacement in the x_1 direction while the tractions t_2 and t_3 are zero.

The types of boundary condition are shown in Fig. 1.

For the type 1 problem, calculation of the M-integral only requires the solutions for *uncracked* composite wedges with the same boundary conditions. Such solutions have been provided by Ting (1982) using an elegant formulation based on Stroh's method (1958) for anisotropic elasticity. It will be shown in the following that for type 2 and 3 problems the

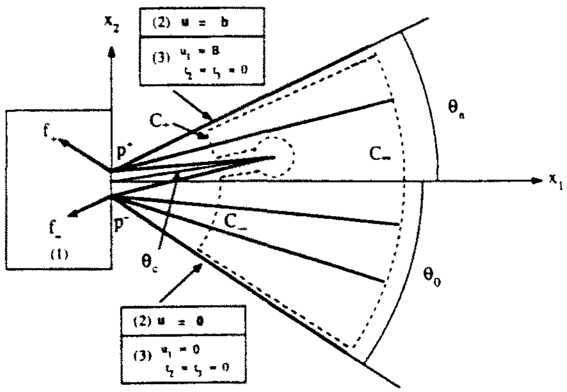

Fig. 1. A composite wedge subjected to three loading conditions.

energy release rates can also be calculated using the solutions for uncracked composite wedges with the same boundary conditions which can be obtained by the formulation given by Ting (1982). For type 2 and type 3 problems, the results show that the energy release rate is independent of the orientation of the crack.

If the edge crack is located in one of the homogeneous wedges, the near-tip stress field is proportional to $r^{-1/2}$ and is characterized by three stress intensity factors corresponding to the traction components along the prolongation of the crack line. The energy release rate is given by a quadratic in the stress intensity factors (Barnett and Asaro, 1972). If the loading, geometry, and materials considered are symmetric or anti-symmetric with respect to the crack line such that only one of the stress intensity factors is nonzero, then the stress intensity factor can be determined from the energy release rate derived in this paper. If the crack lies along one of the interfaces and the crack faces are assumed traction-free, the near-tip stress field exhibits singularities of the types $r^{-1/2}$ and $r^{-1/2 \pm i\gamma}$ in general with γ being a constant related to the elastic constants of the adjoining two materials (Ting, 1986). Bassani and Qu (1989) and Qu and Bassani (1989) discussed cases in which $\gamma = 0$ and the near-tip field is the same as that for homogeneous media. For interface cracks with $\gamma \neq 0$, several definitions of the stress intensity factors have been proposed (Willis, 1971; Suo, 1990; Wu, 1990a) to characterize the near-tip field. The stress intensity factors, regardless of the definition adopted, are related to tractions along the interface in a different manner from those for cracks with a square-root singularity. For traction-free interface cracks, the energy release rate can also be expressed in terms of a quadratic in stress intensity factors (Willis, 1971; Suo, 1990; Wu, 1990a). However, the stress intensity factors and the modes of loading are usually intermixed so that more than one stress intensity factor is present. In this case, knowledge of the energy release rate alone is not sufficient to determine separate stress intensity factors. It is well known that prescription of traction-free conditions generally leads to crack face overlapping (England, 1965; Anderson, 1988; Wu, 1990a). A model to overcome this difficulty was proposed by Comninou (1977) for isotropic materials. The model assumes that frictionless contact zones exist behind the crack tips. It was found for isotropic materials that the near-tip stress field is square-root singular but is characterized by only one independent stress intensity factor relating to the in-plane shear stress for plane strain deformation. The model was extended for anisotropic materials by Wu (1990b). It was shown by Wu (1990b) that in general the singularities of the near-tip stress field for the contact zone model are $r^{-1/2 \pm i\gamma^*}$ where γ^* is a bimaterial constant. The near-tip stress field is characterized by two independent stress intensity factors relating to the in-plane shear and anti-plane shear stresses at the interface and the energy release rate is a quadratic in the stress intensity factors. Conditions for $\gamma^* = 0$ were given by Wu (1990b). For anisotropic bimaterials in which in-plane shear and anti-plane shear deformations are uncoupled, separate stress intensity factors can be obtained from the energy release rate (Wu, 1990b). The energy release rates obtained here are valid for traction-free interface cracks as well as those with contact zones.

The plan of the paper is as follows. In Section 2, the approach using the M-integral to obtain the energy release rates is reviewed. In Section 3, the form of the basic solutions for uncracked composite wedges as presented by Ting (1982) is given and the relevant results for the application of the M-integral approach are derived. Results of the energy release rates are presented in Section 4.

2. THE M-INTEGRAL

The M-integral is defined by

$$M = \int_C (Wn_i x_i - t_k u_{k,i} x_i)\,\mathrm{d}s \tag{3}$$

where W is the strain energy density, n_i is the unit normal to curve C, t_i is the traction on C, and repeated indices imply summation. This summation convention will be followed throughout this paper. The procedure to derive the energy release rates hinges on the fact that $M = 0$ if C is a closed path surrounding a simply connected region in which no elastic singularities are present. If C is a circular arc, eqn (3) can be expressed in a more convenient form in terms of the displacements and the stress functions Φ_i as

$$M = \frac{1}{2}\int \left(\frac{\partial \Phi_i}{\partial r}\frac{\partial u_i}{\partial \theta} + \frac{\partial \Phi_i}{\partial \theta}\frac{\partial u_i}{\partial r}\right) r\,\mathrm{d}\theta. \tag{4}$$

The stress functions are defined such that the traction $\mathbf{t}$ on a contour Γ is given by

$$\mathbf{t} = -\frac{\partial \Phi}{\partial s} \tag{5}$$

where s is the arc length along Γ. If C is a radial line, eqn (3) becomes

$$M = \int \frac{\partial \Phi_i}{\partial r}\frac{\partial u_i}{\partial r}\, r\,\mathrm{d}r. \tag{6}$$

For the problems of interest, let C be the path shown as dotted line in Fig. 1. As discussed by Nachman and Walton (1980), application of the M-integral results in the following identity:

$$G = \frac{1}{a}(M(C_\infty) - M(C_+) - M(C_-)) \tag{7}$$

where C_∞ is a circular arc surrounding the crack and subtending from θ_0 to θ_n, C_- and C_+ are small arcs surrounding the two apexes, p^+ and p^-, subtending from θ_0 to θ_c and θ_c to θ_n, respectively, a is the length of the crack, and G is the energy release rate. For eqn (7) to hold, the contributions to the M-integral as evaluated along the wedge faces and the crack faces must vanish. From eqn (6), this requires

$$t_i \frac{\partial u_i}{\partial r} = 0 \tag{8}$$

or the traction must be orthogonal to the displacement gradient along the radial line. The requirement is satisfied for wedge faces in the problems of interest. The requirement is also satisfied for the crack faces if the crack faces are assumed to be either traction-free or if frictionless contact zones are assumed to exist since the tangential components of the traction and the normal displacement are zero in the contact zones.

If C_∞ is sufficiently far away from the origin, $M(C_\infty)$ can be evaluated using solutions for an uncracked composite wedge with the same boundary conditions. Such solutions are provided by Ting (1982). Similarly, if C_- and C_+ are sufficiently close to the apexes p^+ and p^- of the cracked composite, $M(C_+)$ and $M(C_-)$ can also be evaluated using the asymptotic solutions for sub-wedges with apexes p^+ and p^-, respectively. For type 1 problems, Ting's solution with concentrated force acting at the apex is again applicable. For type 2 and 3 problems, the asymptotic stress field corresponding to homogeneous boundary conditions $\mathbf{t} = \mathbf{0}$ on one face and $\mathbf{u} = \mathbf{0}$ or $u_1 = t_2 = t_3 = 0$ on the other must be sought. In general, the actual form of the asymptotic stress field of an anisotropic composite wedge depends on the elastic constants of the individual homogeneous wedges, geometry of the composite wedge, boundary conditions on the wedge faces, and interface conditions. If the displacement field is required to be continuously bounded at the apex, the asymptotic stress near the apex has the form (Ting, 1981)

$$r^{-\xi}F(r,\theta) \tag{9}$$

where $\xi < 1$ and F is a real function of the polar coordinates (r, θ). The function F contains $\log r$, powers of $\log r$ and trigonometric functions with θ and $\log r$ as arguments. This requirement of bounded displacement is satisfied if the apex of the wedge is not subjected to concentrated forces or the displacements prescribed on the wedge faces are identical and finite at the apex. In fact, Ting's solution, to be discussed in the next section, corresponds to the cases where there are forces acting at the apex or the displacement is discontinuous at the apex. In those cases, the stress singularity is $1/r$. For type 2 and 3 problems, the aforementioned situations are not present, and it is therefore reasonable to *assume* that eqn (9) is valid near the apex. This is true for the homogeneous sub-wedge with wedge angle π for type 2 problems in which case $\xi = 0.5$ (Ting, 1986). It is not difficult to show that for type 3 problems with the same wedge ξ is also 0.5. With the asymptotic stress of eqn (9) substituted in eqn (4), the corresponding $M(C_+)$ and $M(C_-)$ vanish as $r \to 0$.

3. UNCRACKED COMPOSITE WEDGES

For the composite wedge shown in Fig. 1 but without the crack, the solutions in the kth composite for the three types of boundary conditions shown in Fig. 1 can be represented as (Ting, 1982)

$$\mathbf{u}^{(k)} = \frac{\log r}{\pi}\mathbf{h} + \hat{\mathbf{S}}^{(k)}(\theta)\mathbf{h} + \hat{\mathbf{H}}^{(k)}(\theta)\mathbf{g} + \mathbf{c}^{(k)} \tag{10}$$

$$\mathbf{\Phi}^{(k)} = \frac{\log r}{\pi}\mathbf{g} - \hat{\mathbf{L}}^{(k)}(\theta)\mathbf{h} + \hat{\mathbf{S}}^{(k)T}(\theta)\mathbf{g} \tag{11}$$

where $\mathbf{h}$, $\mathbf{g}$ and $\mathbf{c}^{(k)}$ are real constant vectors. In eqns (10) and (11) the matrices are given by

$$\hat{\mathbf{S}}(\theta) = -\frac{1}{\pi}\int_0^\theta \mathbf{T}^{-1}(\omega)\mathbf{R}^T(\omega)\,\mathrm{d}\omega \tag{12}$$

$$\hat{\mathbf{H}}(\theta) = \frac{1}{\pi}\int_0^\theta \mathbf{T}^{-1}(\omega)\,\mathrm{d}\omega \tag{13}$$

$$\hat{\mathbf{L}}(\theta) = \frac{1}{\pi}\int_0^\theta [\mathbf{Q}(\omega) - \mathbf{R}(\omega)\mathbf{T}^{-1}(\omega)\mathbf{R}^T(\omega)]\,\mathrm{d}\omega \tag{14}$$

and

$$Q_{ik}(\omega) = c_{ijks} n_j(\omega) n_s(\omega) \tag{15}$$

$$R_{ik}(\omega) = c_{ijks} n_j(\omega) m_s(\omega) \tag{16}$$

$$T_{ik}(\omega) = c_{ijks} m_j(\omega) m_s(\omega) \tag{17}$$

$$\mathbf{n}^T(\omega) = [\cos\omega, \sin\omega, 0] \tag{18}$$

$$\mathbf{m}^T(\omega) = [-\sin\omega, \cos\omega, 0] \tag{19}$$

with c_{ijks} being the elastic constant. The superscript k in $\hat{\mathbf{S}}^{(k)}$, $\hat{\mathbf{H}}^{(k)}$ and $\hat{\mathbf{L}}^{(k)}$ denotes that the elastic constants for the kth wedge are used. Equations (10) and (11) are valid for isotropic materials as well. The matrices $\hat{\mathbf{S}}$, $\hat{\mathbf{H}}$ and $\hat{\mathbf{L}}$ for isotropic materials are listed below

$$\hat{\mathbf{S}}(\theta) = \frac{1}{\pi(\kappa+1)} \begin{bmatrix} 1-\cos 2\theta & -((\kappa-1)\theta+\sin 2\theta) & 0 \\ (\kappa-1)\theta-\sin 2\theta & \cos 2\theta - 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{20}$$

$$\hat{\mathbf{H}}(\theta) = \frac{1}{2\pi\mu(\kappa+1)} \begin{bmatrix} 2\kappa\theta+\sin 2\theta & 1-\cos 2\theta & 0 \\ 1-\cos 2\theta & 2\kappa\theta-\sin 2\theta & 0 \\ 0 & 0 & 2(\kappa+1)\theta \end{bmatrix} \tag{21}$$

$$\hat{\mathbf{L}}(\theta) = \frac{2\mu}{\pi(\kappa+1)} \begin{bmatrix} 2\theta+\sin 2\theta & 1-\cos 2\theta & 0 \\ 1-\cos 2\theta & 2\theta-\sin 2\theta & 0 \\ 0 & 0 & \frac{(\kappa+1)}{2}\theta \end{bmatrix} \tag{22}$$

where $\kappa = 3-4\nu$ for plane strain deformation with ν being Poisson's ratio and μ the shear modulus.

Note that the constant vectors $\mathbf{h}$ and $\mathbf{g}$ in eqns (10) and (11) are the same for all k. This is because the traction $\mathbf{t}_\theta$ and the displacement gradient $\partial\mathbf{u}/\partial r$ corresponding to eqns (10) and (11) on the radial planes are given by

$$\mathbf{t}_\theta = -\frac{\partial\boldsymbol{\Phi}}{\partial r} = -\frac{1}{\pi r}\mathbf{g} \tag{23}$$

$$\frac{\partial\mathbf{u}}{\partial r} = \frac{1}{\pi r}\mathbf{h} \tag{24}$$

and these quantities are continuous across the interfaces. The rigid body displacement $\mathbf{c}^{(k)}$ in eqn (10) is determined by the continuity conditions on the displacement at the interfaces, i.e.

$$\mathbf{c}^{(k+1)} + \hat{\mathbf{S}}^{(k+1)}(\theta_k)\mathbf{h} + \hat{\mathbf{H}}^{(k+1)}(\theta_k)\mathbf{g} = \mathbf{c}^{(k)} + \hat{\mathbf{S}}^{(k)}(\theta_k)\mathbf{h} + \hat{\mathbf{H}}^{(k)}(\theta_k)\mathbf{g} \tag{25}$$

for $k = 1, 2, \ldots, n-1$. When $\mathbf{h}$ and $\mathbf{g}$ are known, eqn (25) provides solutions for $\mathbf{c}^{(2)}, \ldots, \mathbf{c}^{(n)}$ in terms of $\mathbf{c}^{(1)}$ which can be arbitrary.

With the form of the solutions represented as in eqns (10) and (11), the condition that the contribution to the M-integral along the radial lines vanishes as described by eqn (8) is expressed by

$$\mathbf{g}^T\mathbf{h} = 0. \tag{26}$$

The contribution to the M-integral, M_k, along part of the circular arc in the kth wedge can be obtained by substituting eqns (10) and (11) into eqn (4). The result is given by

$$M_k = \frac{1}{2\pi}[\mathbf{g}^T\Delta\mathbf{u}^{(k)} + \mathbf{h}^T\Delta\mathbf{\Phi}^{(k)}] \tag{27}$$

where

$$\Delta\mathbf{u}^{(k)} = \mathbf{u}^{(k)}(\theta_k) - \mathbf{u}^{(k)}(\theta_{k-1}) \tag{28}$$

$$\Delta\mathbf{\Phi}^{(k)} = \mathbf{\Phi}^{(k)}(\theta_k) - \mathbf{\Phi}^{(k)}(\theta_{k-1}). \tag{29}$$

In deriving eqn (27), eqn (26) has been used. The expression of the M-integral for the entire circular subtending from θ_0 to θ_n is simply

$$M = \sum_{k=1}^{n} M_k = \frac{1}{2\pi}[\mathbf{g}^T\mathbf{b} + \mathbf{h}^T\mathbf{f}] \tag{30}$$

where

$$\mathbf{b} = \mathbf{u}^{(n)}(r, \theta_n) - \mathbf{u}^{(1)}(r, \theta_0) \tag{31}$$

$$\mathbf{f} = \mathbf{\Phi}^{(n)}(r, \theta_n) - \mathbf{\Phi}^{(1)}(r, \theta_0). \tag{32}$$

Equation (30) is the main result of this section. The simple form of eqn (30) enables one to obtain the value of the M-integral as soon as the constants $\mathbf{h}$ and $\mathbf{g}$ are determined.

4. ENERGY RELEASE RATES

In this section explicit solutions of the energy rates for the problems shown in Fig. 1 are derived using eqn (7) in conjunction with eqn (30). Without loss of generality, the crack is assumed to lie in the kth wedge, i.e. $\theta_{k-1} \leqslant \theta_c \leqslant \theta_k$.

4.1. *Problem* 1

Consider the line force $\mathbf{f}_+$ acting at the apex of the composite wedge subtending from θ_c to θ_n in Fig. 1. As the traction is zero on the wedge faces, from eqn (23), it follows that

$$\mathbf{g} = 0. \tag{33}$$

The constant $\mathbf{h}$ can be determined by the fact the traction resultant on the circular arc from θ_c to θ_n must be in equilibrium with the applied force $\mathbf{f}_+$, i.e.

$$\sum_{j=k+1}^{n} \Delta\mathbf{\Phi}^{(j)} + \mathbf{\Phi}^{(k)}(r, \theta_k) - \mathbf{\Phi}^{(k)}(r, \theta_c) = \mathbf{f}_+. \tag{34}$$

From eqn (11), $\mathbf{h}$ is determined as

$$\mathbf{h} = -\mathscr{L}_+^{-1}\mathbf{f}_+ \tag{35}$$

where $\mathscr{L}_+$ is defined by

$$\mathscr{L}_+ = \sum_{j=k+1}^{n} \Delta\hat{\mathbf{L}}^{(j)} + \hat{\mathbf{L}}^{(k)}(\theta_k) - \hat{\mathbf{L}}^{(k)}(\theta_c). \tag{36}$$

With eqns (33) and (35), from eqn (30), the corresponding $M(C_+)$ is

$$M(C_+) = -\frac{1}{2\pi}\mathbf{f}_+^T \mathscr{L}_+^{-1}\mathbf{f}_+. \tag{37}$$

Similarly $M(C_-)$ for the composite wedge $\theta_0 \leqslant \theta \leqslant \theta_c$ with the line force $\mathbf{f}_-$ acting at the apex is given by

$$M(C_-) = -\frac{1}{2\pi}\mathbf{f}_-^T \mathscr{L}_-^{-1}\mathbf{f}_- \tag{38}$$

where $\mathscr{L}_-$ is defined by

$$\mathscr{L}_- = \sum_{j=1}^{k-1}\Delta\hat{\mathbf{L}}^{(j)} + \hat{\mathbf{L}}^{(k)}(\theta_c) - \hat{\mathbf{L}}^{(k)}(\theta_{k-1}). \tag{39}$$

For $M(C_\infty)$, the solutions of the entire composite wedge from $\theta_0 \leqslant \theta \leqslant \theta_n$ with the force $\mathbf{f}_+ + \mathbf{f}_-$ must be used. The result is

$$M(C_\infty) = -\frac{1}{2\pi}(\mathbf{f}_+ + \mathbf{f}_-)^T \mathscr{L}^{-1}(\mathbf{f}_+ + \mathbf{f}_-) \tag{40}$$

where $\mathscr{L}$ is given by

$$\mathscr{L} = \mathscr{L}_+ + \mathscr{L}_- = \sum_{j=1}^{n}\Delta\hat{\mathbf{L}}^{(j)}. \tag{41}$$

Substitution of eqns (37), (38) and (40) into eqn (7) leads to

$$G = \frac{1}{2\pi a}[\mathbf{f}_+^T \mathscr{L}_+^{-1}\mathbf{f}_+ + \mathbf{f}_-^T \mathscr{L}_-^{-1}\mathbf{f}_- - (\mathbf{f}_+ + \mathbf{f}_-)^T \mathscr{L}^{-1}(\mathbf{f}_+ + \mathbf{f}_-)]. \tag{42}$$

For homogeneous isotropic wedges ($n = 1$), eqn (42) reduces to the result in Freund (1978). For bimaterial isotropic wedges with the crack lying at the interface ($n = 2$, $\theta_c = \theta_1 = 0$), eqn (42) gives the result in Nachman and Walton (1980). For the special case of a semi-infinite crack penetrating the interface of an isotropic bimaterial and subjected to a pair of self-equilibrating line forces, $\mathbf{F}$ and $-\mathbf{F}$, at the interface as shown in Fig. 2, the energy release rate is

$$G = \frac{\pi(1-\alpha)(\kappa+1)}{4(\pi^2 - 4\alpha^2)a\mu}(F_1^2 + F_2^2) + \frac{2F_3^2}{\pi a(\mu+\mu')} \tag{43}$$

where α is one of the two Dundurs bimaterial parameters given by

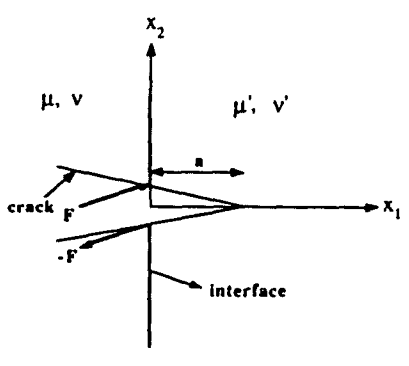

Fig. 2. A semi-infinite crack penetrating the interface of an isotropic bimaterial subjected to a pair of self-equilibrating line forces.

$$\alpha = \frac{\mu'(\kappa+1)-\mu(\kappa'+1)}{\mu'(\kappa+1)+\mu(\kappa'+1)}. \tag{44}$$

Equation (43) is a more general result than that given in Kubo (1982) where only the anti-plane shear loading F_3 is considered.

4.2. *Problem 2*

In problem 2, the wedge faces $\theta = \theta_0, \theta_n$ are subjected to constant displacements such that

$$\mathbf{u}^{(n)}(r,\theta_n)-\mathbf{u}^{(1)}(r,\theta_0) = \mathbf{b}. \tag{45}$$

As discussed in Section 2, if the displacement is assumed to be continuously bounded at the apex, both $M(C_+)$ and $M(C_-)$ vanish in the limit as $r \to 0$.

For $M(C_\infty)$, the solution for the uncracked composite wedge with eqn (45) is derived as follows. Since the displacements at the wedge faces are constant, it follows from eqn (24) that

$$\mathbf{h} = \mathbf{0}. \tag{46}$$

The constant $\mathbf{g}$ is determined by eqn (45). The result is

$$\mathbf{g} = \mathscr{H}^{-1}\mathbf{b} \tag{47}$$

where

$$\mathscr{H} = \sum_{k=1}^{n} (\hat{\mathbf{H}}^{(k)}(\theta_k)-\hat{\mathbf{H}}^{(k)}(\theta_{k-1})). \tag{48}$$

Substituting eqns (46) and (47) into eqn (30) yields

$$G = \frac{1}{2\pi a}\mathbf{b}^T\mathscr{H}^{-1}\mathbf{b}. \tag{49}$$

Note that in eqn (49) θ_c is not present. This leads to a remarkable conclusion that the energy release rate is independent of the inclination of the crack.

For a homogeneous ($n = 1$) isotropic wedge, eqn (49) agrees with the result in Ouchterlony (1980). Another interesting case is that of an infinite composite strip with a semi-infinite crack subjected to constant displacements as shown in Fig. 3. The energy release

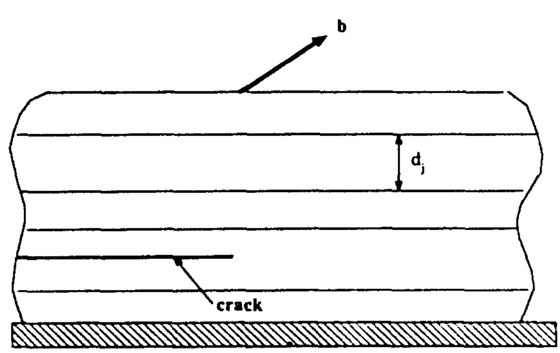

Fig. 3. A composite strip with clamped boundary conditions.

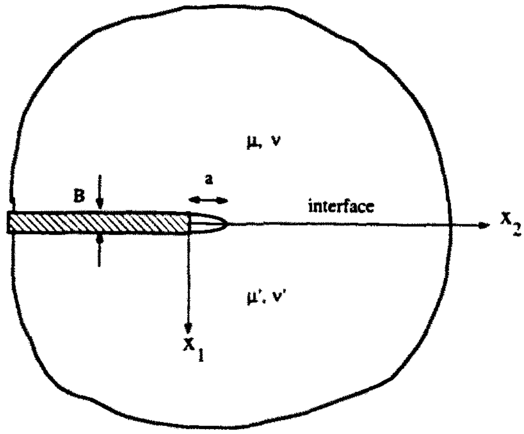

Fig. 4. An interface crack in an isotropic bimaterial opened by a frictionless wedge.

rate in this case can be obtained by letting $a \to \infty$ and $\theta_j \to 0$, $j = 0, 1, \ldots, n$, such that $a(\theta_j - \theta_{j-1}) = d_j$ with d_j being the thickness of the jth layer. The resulting expression of $\mathscr{K}$ can be shown to be

$$\lim_{a\to\infty} a\mathscr{K} = \frac{1}{\pi}\sum_{j=1}^{n} (\mathbf{T}^{(j)})^{-1} d_j \tag{50}$$

where $\mathbf{T} = \mathbf{T}(0)$ in eqn (17). The corresponding energy release rate is thus given by

$$G = \frac{1}{2}\mathbf{b}^T\left(\sum_{j=1}^{n} (\mathbf{T}^{(j)})^{-1} d_j\right)^{-1}\mathbf{b}. \tag{51}$$

The result for $n = 1$ for isotropic materials has been given by Rice (1968). In fact, eqn (51) could also be obtained by the J-integral in a similar manner as discussed in Rice (1968).

4.3. *Problem* 3

Problem 3 is characterized by the following boundary conditions

$$u_1^{(n)}(r, \theta_n) - u_1^{(1)}(r, \theta_0) = B \tag{52}$$

$$t_2 = t_3 = 0 \quad \text{at} \quad \theta = \theta_0, \theta_n \tag{53}$$

where $u_1^{(n)}(r, \theta_n)$ and $u_1^{(1)}(r, \theta_0)$ are constants. As discussed in Section 2, $M(C_+) = M(C_-) = 0$ in the limit as $r \to 0$ if the displacement is assumed to be continuously bounded at the apex. For $M(C_\infty)$, the relevant solution is that for an uncracked wedge subjected to the boundary conditions eqns (52) and (53). Since the displacement u_1 and the tractions t_2, t_3 are constant at the wedge faces, it follows from eqns (23) and (24) that $h_1 = g_2 = g_3 = 0$. The remaining constants g_1, h_2 and h_3 in eqns (10) and (11) are determined by considering eqn (52) and the fact that the tractions t_2 and t_3 must vanish on a circular arc $\theta_0 \leqslant \theta \leqslant \theta_n$. The result is

$$\begin{bmatrix} g_1 \\ h_2 \\ h_3 \end{bmatrix} = \mathscr{Y}^{-1}\begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix} \tag{54}$$

where $\mathscr{Y}$ is given by

$$\mathscr{Y}^{*} = \sum_{i=1}^{n} \Delta\mathbf{V}^{(i)} \tag{55}$$

and $\Delta\mathbf{V}^{(i)} = \mathbf{V}^{(i)}(\theta_i) - \mathbf{V}^{(i)}(\theta_{i-1})$ with $\mathbf{V}$ defined by

$$\mathbf{V} = \begin{bmatrix} \hat{H}_{11} & \hat{S}_{12} & \hat{S}_{13} \\ \hat{S}_{12} & -\hat{L}_{22} & -\hat{L}_{23} \\ \hat{S}_{13} & -\hat{L}_{23} & -\hat{L}_{33} \end{bmatrix}. \tag{56}$$

From eqn (30), the value of $M(C_x)$ is simply given by

$$M(C_x) = \frac{1}{2\pi} g_1 B. \tag{57}$$

The corresponding energy release rate is

$$G = \frac{1}{2\pi a} (\mathscr{Y}^{*-1})_{11} \mathrm{B}^2. \tag{58}$$

It should be noted that eqn (58) is also independent of the crack angle θ_c.

For a semi-infinite interface crack in an infinite isotropic bimaterial opened by a frictionless wedge as shown in Fig. 4, eqn (58) becomes

$$G = \frac{\mu B^2}{2\pi a} \frac{\kappa' + 1 + \Gamma(\kappa + 1)}{(\kappa + 1/\Gamma)(\kappa' + \Gamma)} \tag{59}$$

where $\Gamma = \mu'/\mu$. For homogeneous materials, eqn (59) is in agreement with that in Freund (1978).

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